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Channel flow with temperature-dependent viscosity and internal viscous dissipation

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This paper uses asymptotic methods to analyse the flow in a narrow channel of a fluid with temperature-dependent viscosity and internal viscous dissipation. When the Nahme–Griffith number is large we show how the flow evolves from Poiseuille flow with a uniform temperature distribution to a plug flow with hot boundary layers on the walls. An asymptotic solution is obtained for the flow in the region of transition from Poiseuille to plug flow and an explicit equation is derived for the pressure gradient in terms of the local downstream co-ordinate in this transition region.

1. Introduction

The flow of fluids with temperature-dependent viscosity has been considered by many authors. This paper follows the approximations used by Pearson (1977) who explains the motivation for these approximations and their limits. We use the same equations for a two-dimensional steady flow but consider the evolution of the flow as an entry problem. The approach used is similar to that of Ockendon & Ockendon (1977) who considered the problem in which viscous dissipation is negligible and temperature changes within the flow are caused by heating or cooling the walls of the channel. In the present case the temperature of the walls T_0 is the same as the temperature of the inlet flow and the subsequent temperature variations are caused by internal viscous heating.

There are three basic length scales: the width $2d$ of the channel, the mechanical relaxation length l_1 and the thermal relaxation length l_2. If the average fluid velocity at entry is $\frac{2}{3}U$ and the dynamic viscosity of the fluid at temperature T_0 is μ_0, then $l_1 = Ud^2\rho/\mu_0$. On the assumption that the fluid has constant density ρ, specific heat c and thermal conductivity k we can define $l_2 = Ud^2\rho c/k$.

The following work depends on the assumption that the Prandtl number, l_2/l_1, is large so that we can reasonably assume a fully developed Poiseuille flow at constant temperature T_0 at the inlet $x = 0$. It is also assumed that the Péclet number, l_2/d, is large so that a lubrication approximation is valid throughout the flow. We take the viscosity in the form $\mu_0 \exp[-b(T^* - T_0)]$, where T^* is the temperature and b is constant. The third dimensionless quantity which then arises is the Nahme–Griffith number, $\beta = \mu_0 U^2 b/k$; we assume that β is large and base our asymptotic analysis on this parameter.

2. Equations

The variables are non-dimensionalized by using dy as the transverse co-ordinate and $l_2 x$ as the co-ordinate measured in the downstream direction. We introduce a stream function $Uh\psi$ and write pressure as $(\mu_0 Ul_2/d^2)p$, viscosity as $\mu_0\mu$ and temperature as $T_0+\mu_0 U^2T/k$. On using the lubrication approximations the downstream momentum equation is

$$\mu\, \partial^2\psi/\partial y^2 = yp', \tag{2.1}$$

where $p' = dp/dx$ and the energy equation is

$$\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y}=\frac{\partial^2 T}{\partial y^2}+\mu\left(\frac{\partial^2\psi}{\partial y^2}\right)^2, \tag{2.2}$$

where $\mu = e^{-\beta T}$.

The boundary conditions at the inlet $x = 0$ are

$$T = 0, \quad dp/dx = -1, \quad \psi = -\tfrac{1}{6}y^3+\tfrac{1}{2}y \tag{2.3}$$

and, on the walls $y = \pm 1$,

$$T = 0, \quad \frac{\partial\psi}{\partial y}=\frac{\partial\psi}{\partial x}=0. \tag{2.4}$$

The non-dimensionalization used here has been chosen so that the pressure gradient is normalized with respect to its value at the inlet. The relationships between these variables and those used by Pearson (1977) are listed in the appendix.

3. Boundary-layer development

The early development of the flow is similar to that described for the non-dissipative case by Ockendon & Ockendon (1977). The flow in the central core remains the same as at entry and the temperature only begins to change appreciably in thermal boundary layers which are of thickness $O(x^{\frac{1}{3}})$. Within these layers there is a similarity solution obtained by writing

$$\psi = -\tfrac{1}{3}+x^{\frac{2}{3}}f(\zeta), \quad T = x^{\frac{2}{3}}g(\zeta), \quad \zeta = (y+1)/x^{\frac{1}{3}}$$

and hence obtaining the equations

$$\mu f'' = 1 \tag{3.1}$$

and

$$\tfrac{2}{3}(f'g-fg') = g''+\mu. \tag{3.2}$$

As long as $x^{\frac{2}{3}}\beta \ll 1$, $\mu = 1$ to first order and we can solve equation (3.1) to give $f = \frac{1}{2}\zeta^2$ and then (3.2) becomes

$$g''+\tfrac{1}{3}\zeta^2 g' - \tfrac{2}{3}\zeta g = -1. \tag{3.3}$$

To obtain the correct matching condition for g as $\zeta\to\infty$ we solve the energy equation in the core. Since x and T are both small the conduction term may be neglected and we obtain

$$T = 2xy^2/(1-y^2), \tag{3.4}$$

thus showing that the temperature in the core grows linearly with x. The boundary conditions for (3.3) are therefore

$$g(0) = 0 \quad \text{and} \quad g(\zeta)\sim 1/\zeta \quad \text{as} \quad \zeta\to\infty.$$

The similarity solution breaks down when the temperature in the boundary layer becomes sufficiently large for the viscosity variation to affect the first-order flow. This occurs when $x = O(\beta^{-\frac{3}{2}})$. The equations in the boundary layer are then obtained by writing $x = \beta^{-\frac{3}{2}}\bar{x}$, $y = -1+\beta^{-\frac{1}{2}}\bar{y}$, $T = \beta^{-1}\bar{T}$ and $\psi = -\frac{1}{3}+\beta^{-1}\bar{\psi}$ to give

$$\partial^2\bar{\psi}/\partial\bar{y}^2 = e^{\bar{T}} \tag{3.5a}$$

and

$$\frac{\partial\bar{\psi}}{\partial\bar{y}}\frac{\partial\bar{T}}{\partial\bar{x}}-\frac{\partial\bar{\psi}}{\partial\bar{x}}\frac{\partial\bar{T}}{\partial\bar{y}} = \frac{\partial^2\bar{T}}{\partial\bar{y}^2}+e^{\bar{T}}. \tag{3.5b}$$

The boundary conditions at the wall are

$$\bar{\psi} = 0, \quad \frac{\partial\bar{\psi}}{\partial\bar{y}} = 0, \quad \bar{T} = 0 \quad \text{when} \quad \bar{y} = 0 \tag{3.6}$$

and matching with the Poiseuille flow in the core gives

$$\bar{\psi} \sim \tfrac{1}{2}\bar{y}^2 \quad \text{and} \quad \bar{T} \sim \bar{x}/\bar{y} \quad \text{as} \quad \bar{y}\to\infty. \tag{3.7}$$

To determine the solution of these equations completely it is also necessary to match with the similarity solution of (3.3) as $\bar{x}\to 0$.

These boundary-layer equations do not have a similarity solution. However, they do bear a resemblance to equations of the form $\partial T/\partial t = \partial^2T/\partial y^2+e^T$ with $T = 0$ on $y = 0$, a. There is no steady-state solution of this problem if a is greater than some critical value a_c and it has been shown by Fujita (1969) that the solution of the unsteady equation will blow up in finite time if $a > a_c$. A rigorous proof of blow-up for equations (3.5) and (3.6) has not been established but asymptotic methods and physical intuition support the hypothesis that $\bar{T}$ becomes infinitely large as $\bar{x}$ approaches a finite value k. We assume that such a breakdown does occur and sketch how the boundary-layer solution of (3.5) splits up asymptotically as $\bar{x}\to k$. The basic structure of the boundary layers in this limit is the same as that found by Pearson (1977) in § 4.1 for his similarity solution. This structure is illustrated in figure 1.

We let the maximum value of $\bar{T}$ for a particular value of $\bar{x}$ be σ where $\sigma \gg 1$ and suppose that this maximum occurs at $\bar{y} = \epsilon_1$. We expect that the dominant effects near this maximum of the temperature will be dissipation and conduction. To analyse the region we put

$$\bar{T} = \sigma+\hat{T}, \quad \bar{y} = \epsilon_1+\epsilon_2\hat{y}, \quad \bar{x} = k+\zeta$$

and consider the structure of the solution of equations (3.5) as $\zeta\to 0$ from below. Then, from (3.5a) we see that $\bar{\psi} = O(\epsilon_2^2 e^{\sigma})$ and putting $\bar{\psi} = \epsilon_2^2 e^{\sigma}\hat{\psi}$ in (3.5b) we obtain

$$\epsilon_2^3 e^{\sigma}\left(\frac{\partial\hat{\psi}}{\partial\hat{y}}\frac{\partial\hat{T}}{\partial\bar{x}}-\frac{\partial\hat{\psi}}{\partial\bar{x}}\frac{\partial\hat{T}}{\partial\hat{y}}\right) = \frac{\partial^2\hat{T}}{\partial\hat{y}^2}+\epsilon_2^2 e^{\sigma}e^{\hat{T}}.$$

Thus we need to take $\epsilon_2 = e^{-\frac{1}{2}\sigma}$ to get the expected balance in this region. We can choose σ and ϵ_1 so that $\hat{T} = \partial\hat{T}/\partial\hat{y} = 0$ at $\hat{y} = 0$ and then

$$\hat{T} = -2\log\cosh(\hat{y}/\sqrt{2}) \quad \text{and} \quad \hat{\psi} = 2\log\cosh(\hat{y}/\sqrt{2})+A\hat{y}+B.$$

As $\hat{y}\to-\infty$ we approach the wall and there is a region in which conduction dominates. In this region, $\bar{T} \simeq \sigma+\sqrt{2}\,\hat{y}$ and using the wall boundary condition on $\bar{T}$ gives $\epsilon_1 = 2^{-\frac{1}{2}}\sigma\epsilon_2 = 2^{-\frac{1}{2}}\sigma e^{-\frac{1}{2}\sigma}$. Also the wall boundary conditions on $\bar{\psi}$ determine $A = \sqrt{2}$ and $B = -2\log 2$. Hence as $\hat{y}\to\infty$,

$$\hat{T} \sim -\sqrt{2}\hat{y} \quad \text{and} \quad \hat{\psi} \sim 2\sqrt{2}\hat{y}.$$

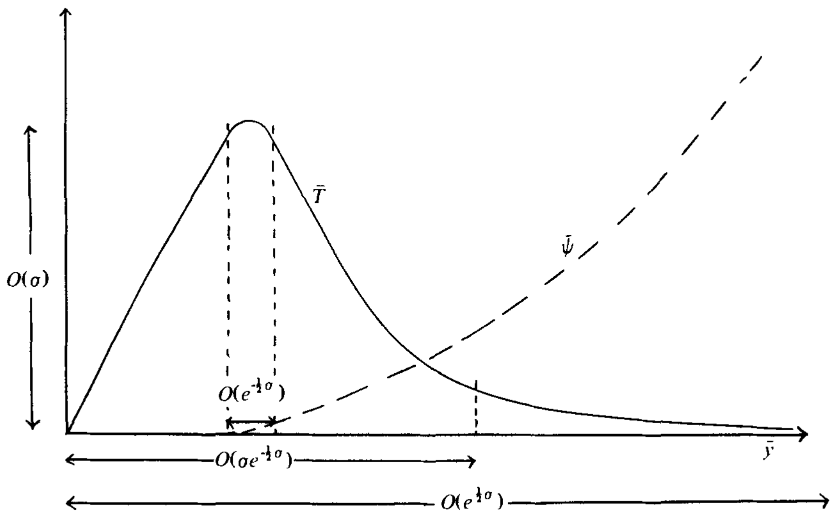

FIGURE 1. Boundary-layer structure as $\bar{x} \to k$.

In the next region we therefore put

$$\hat{y} = \sigma\tilde{y} \quad \text{or} \quad \bar{y} = \sigma e^{-\frac{1}{2}\sigma}(1/\sqrt{2}+\tilde{y}),$$
$$\bar{T} = \sigma\tilde{T} \quad \text{and} \quad \bar{\psi} = \sigma\tilde{\psi}.$$

The dissipation terms are now negligible since $\tilde{T} < 1$ and we have a constant velocity layer in which $\tilde{\psi} = 2\sqrt{2}\tilde{y}$. We can make convection and conduction balance in the energy equation only if $|\zeta| = O(\sigma^2 e^{-\frac{1}{2}\sigma})$ and $\tilde{T}$ then satisfies a diffusion equation. We now need yet another layer in which the velocity can adjust to match with the core flow. Using the fact that $\bar{\psi} \sim \frac{1}{2}\bar{y}^2$ as $\bar{y} \to \infty$ we deduce that $\bar{y} = O(e^{\frac{1}{2}\sigma})$ in this layer.

Thus the boundary layer consists of an 'inner' temperature-dominated layer in which $1+y = O(e^{-\frac{1}{2}\sigma}\beta^{-\frac{1}{2}})$ when $\zeta = O(\sigma^2 e^{-\frac{1}{2}\sigma})$ and so the width of this layer decreases as $\bar{x} \to k$. At the same time, the maximum value of $\bar{T}$ is $-2\log|\zeta| + 4\log|\log|\zeta|| + O(1)$ which grows without limit as $|\zeta| \to 0$. This inner layer is contained in an 'outer' boundary layer in which the velocity adjusts and the width of this outer layer grows like $O(\beta^{-\frac{1}{2}} e^{\frac{1}{2}\sigma})$ as $\sigma \to \infty$. It is clear that this solution ceases to be valid when the outer boundary layer has spread right across the channel, which happens when $\sigma = \log\beta$. Thus the next region of interest occurs when $k-\bar{x} = O(\beta^{-\frac{1}{2}}(\log\beta)^2)$. Detailed matching of the above asymptotic solution with the core flow confirms that $p'+1$ becomes $O(1)$ when $|\bar{x}-k| = O(\beta^{-\frac{1}{2}}(\log\beta)^2)$. We therefore now use this scaling and concentrate on finding the solution as the critical point $\bar{x} = k$ is approached.

The value of k can only be found by the complete numerical integration of equations (3.5) with the given boundary conditions. Since we can only attempt an analysis of these equations asymptotically when the maximum value of $\bar{T}$ is large, our approach will never determine k analytically.

4. Solution when $p'+1 = O(1)$

(*a*) *Core flow.* We now write $x = \beta^{-\frac{3}{2}}\bar{x} = \beta^{-\frac{3}{2}}(k+\beta^{-\frac{1}{2}}(\log\,\beta)^2\tilde{x})$ and assume that $p'+1 = O(1)$. The core flow is now slipping relative to the walls and in the central core

$$\psi = \tfrac{1}{6}p'y^3-\tfrac{1}{6}p'y+\tfrac{1}{3}y, \tag{4.1}$$

so that $\partial\psi/\partial y = \frac{1}{3}(p'+1)$ on $y = \pm 1$. From (3.4) we see that the temperature is $O(\beta^{-\frac{3}{2}})$ and so satisfies the equation

$$\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial\tilde{x}}-\frac{\partial\psi}{\partial\tilde{x}}\frac{\partial T}{\partial y} = 0$$

to first order. Hence $T = \beta^{-\frac{3}{2}}f(\psi)$. As $\tilde{x}\to-\infty$, $p'\to-1$ and $T\to\beta^{-\frac{3}{2}}\,2ky^2/(1-y^2)$, which determines f by the implicit formula

$$\psi = -\frac{(3k+f(\psi))\,(f(\psi))^{\frac{1}{2}}}{3(2k+f(\psi))^{\frac{3}{2}}}. \tag{4.2}$$

As $y\to-1$ we now need a boundary layer to adjust the velocity and to contain the thermal effects. The limiting values of ψ, T as $y\to-1$ are needed to match with the boundary-layer solution. From (4.1) and (4.2) these values are

$$\psi \sim -\tfrac{1}{3}+\tfrac{1}{3}(p'+1)\,(y+1)+O((y+1)^2) \tag{4.3}$$

and

$$T \sim \beta^{-\frac{3}{2}}\left\{\frac{k\sqrt{3}}{\sqrt{2}\,(p'+1)^{\frac{1}{2}}(1+y)^{\frac{1}{2}}}+O((y+1)^{\frac{1}{2}})\right\}. \tag{4.4}$$

(*b*) *Boundary-layer structure.* The boundary-layer breakdown described in §3 indicates that there will be a boundary layer of width $O(\beta^{-1}(\log\,\beta))$ when $\tilde{x} = O(1)$. We therefore write

$$y = -1+\beta^{-1}(\log\beta)\,(\alpha+y_1),$$

where α is chosen to make $y_1 = 0$ at the maximum value of T. We also write

$$\psi = -\tfrac{1}{3}+\beta^{-1}(\log\beta)\,\psi_1(\tilde{x}, y_1)$$

and

$$T = \beta^{-1}(\log\beta)\,T_1(\tilde{x}, y_1).$$

The stream function is now of the right order of magnitude to match the core flow but, as we shall see later, we need to consider another internal constant-velocity layer in order to make the temperature match with its value in the core.

The equations in this boundary layer are thus

$$\frac{\partial^2\psi_1}{\partial y_1^2} = -p'\,\beta^{T_1-1}\log\beta \tag{4.5}$$

and

$$\frac{\partial\psi_1}{\partial y_1}\frac{\partial T_1}{\partial\tilde{x}}-\frac{\partial\psi_1}{\partial\tilde{x}}\frac{\partial T_1}{\partial y_1}-\frac{\partial^2 T_1}{\partial y_1^2} = p'^2\,(\log\beta)\,\beta^{T_1-1}. \tag{4.6}$$

As long as $T_1 < 1$, the right-hand sides of both these equations are negligible and on matching with (4.3) we have, from (4.5),

$$\psi_1 = \tfrac{1}{3}(p'+1)\,(y_1+c_1) \tag{4.7}$$

for $y_1 > 0$, where c_1 is an arbitrary function of $\tilde{x}$. In (4.6) this gives

$$\tfrac{1}{3}(p'+1)\frac{\partial T_1}{\partial \tilde{x}} - \frac{1}{3}\left(p''(y_1+c_1)+p'\frac{dc_1}{d\tilde{x}}\right)\frac{\partial T_1}{\partial y_1} = \frac{\partial^2 T_1}{\partial y_1^2}, \tag{4.8}$$

where $p'' = dp'/d\tilde{x}$.

The above approximation breaks down as $T_1 \to 1$. We may choose α so that $T_1 \to 1$ as $y_1 \to 0$ and then rescale near $y_1 = 0$ by putting

$$y_1 = \frac{1}{\log\beta}y_2$$

and

$$T_1 = 1 + \frac{1}{\log\beta}T_2(\tilde{x}, y_2).$$

It can now be seen that in order to obtain a non-trivial solution for ψ which is capable of satisfying the no-slip condition on the wall we must take $c_1 = 0$ in (4.7). We then write

$$\psi_1 = \frac{1}{\log\beta}\psi_2(\tilde{x}, y_2)$$

so that equations (4.5) and (4.6) become

$$\frac{\partial^2\psi_2}{\partial y_2^2} = -p'\exp[T_2],$$

$$\frac{\partial^2 T_2}{\partial y_2^2} + p'^2\exp[T_2] = 0 \tag{4.9}$$

on neglecting terms of $O((\log\beta)^{-2})$. We solve equations (4.9) using $\partial T_2/\partial y_2 = 0$ at $y_2 = 0$ and ψ_2, $\partial\psi_2/\partial y_2 \to 0$ as $y_2 \to -\infty$ to obtain

$$T_2 = 2C - 2\log\cosh\left(\frac{-p'y_2}{\sqrt{2}\,e^{-C}}\right) \tag{4.10}$$

and

$$\psi_2 = \frac{2}{-p'}\log\cosh\left(\frac{-p'y_2}{\sqrt{2}\,e^{-C}}\right) + \sqrt{2}\,e^C\,y_2 + \frac{2\log 2}{-p'},$$

where C is an arbitrary function of $\tilde{x}$. Matching with (4.7) as $y_2 \to \infty$ implies that

$$C = \log((p'+1)/6\sqrt{2}).$$

By using the condition on the wall that $T_1 = 0$ when $y_1 = -\alpha$ we can determine $\alpha = 6/[(-p')(p'+1)]$.

We now have to find a solution of (4.8) with $c_1 = 0$ which matches with (4.10) as $y_1 \to 0$. This matching condition leads to

$$T_1 \sim 1 + \tfrac{1}{6}p'(p'+1)y_1 \quad \text{as } y_1 \to 0 \tag{4.11}$$

and we must also have that

$$T_1 \to 0 \quad \text{as} \quad y_1 \to \infty.$$

It can now be seen that equation (4.8) with boundary conditions (4.11) has a similarity solution if we write

$$T_1 = g(\eta) \quad \text{where} \quad \eta = (-p')(p'+1)y_1.$$

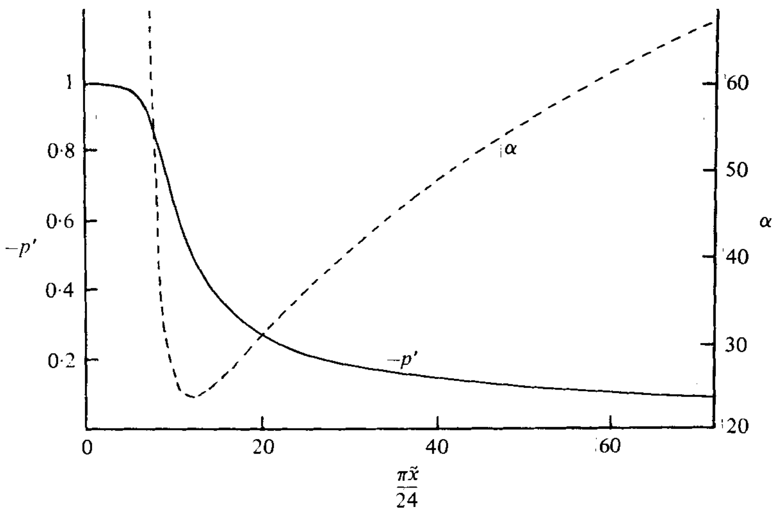

FIGURE 2. Variation of pressure gradient with $\tilde{x}$.

The equation satisfied by g is

$$g'' = \frac{p''}{p'^3(p'+1)}\frac{\eta g'}{3}, \tag{4.12}$$

with $g \sim 1-\frac{1}{6}\eta$ as $\eta \to 0$ and $g \to 0$ as $\eta \to \infty$. Thus a similarity solution is possible if

$$\frac{p''}{p'^3(p'+1)} = -\lambda, \tag{4.13}$$

where λ is some constant. We can then solve equation (4.12) for g to obtain

$$g = 1-\frac{1}{6}\int_0^\eta \exp\left[-\tfrac{1}{144}\pi\tau^2\right]d\tau \tag{4.14}$$

and $\lambda = \frac{1}{24}\pi$. Thus equation (4.13) can be integrated to give

$$-\frac{\pi}{24}\tilde{x} + \text{const.} = \log\left(\frac{-p'}{1+p'}\right) + \frac{1}{p'} - \frac{1}{2p'^2}. \tag{4.15}$$

This expression for p' covers the transition from $p' \to -1$ as $\tilde{x} \to -\infty$ to $p' \to 0$ as $\tilde{x} \to +\infty$. It is plotted in figure 2.

The terms neglected in deriving (4.9) are of $O((\log\beta)^{-2})$ which is small only if β is very large. However, within these limits we have obtained an analytic expression for the pressure gradient in (4.15). Unfortunately the constant of integration in (4.15) can only be determined by matching with the solution of (3.5) which we have been unable to solve analytically. Thus we are still not in a position to determine the pressure, $\int_0^x p'dx$, without recourse to numerical methods.

The final step necessary to complete this part of the solution is to match the temperature T_1 with the temperature in the core region. As $y_1 \to \infty$,

$$T_1 = g \sim \frac{12}{\pi\eta}\exp\left(-\frac{\pi\eta^2}{144}\right), \tag{4.16}$$

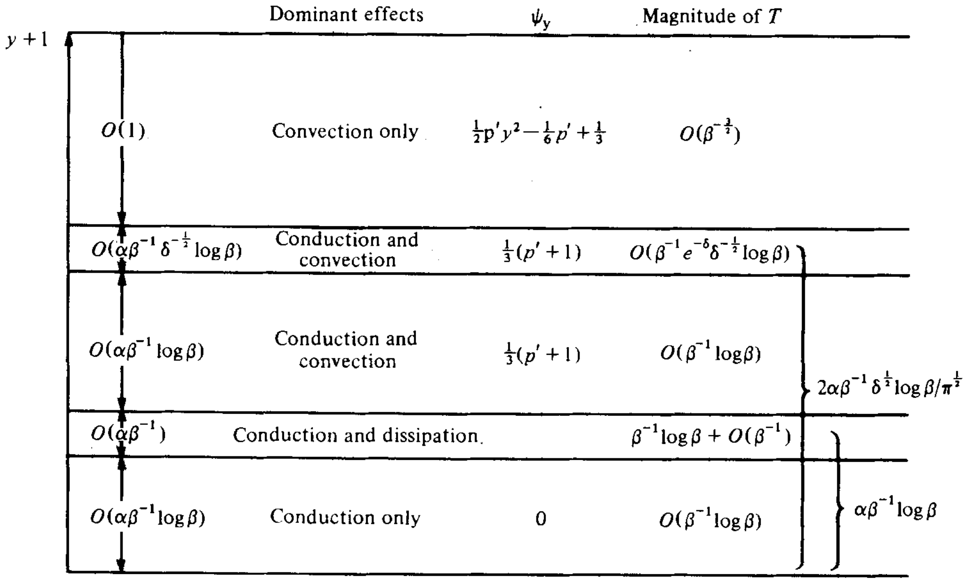

FIGURE 3. Boundary-layer structure when $\tilde{x}-k = O(\beta^{-\frac{1}{2}}(\log\beta)^2)$.

and we need to introduce a narrow free layer within which the velocity is constant but the temperature adjusts so that we can match with the algebraic form of the temperature given by (4.4). We put $\eta = 12\pi^{-\frac{1}{2}}(\delta^{\frac{1}{2}}+z\delta^{-\frac{1}{2}})$, where δ is a large parameter and z is the new variable in the y direction. Using (4.16) we write $T_1 = e^{-\delta}\delta^{-\frac{1}{2}}\theta$, where $\theta \to \pi^{-\frac{1}{2}}e^{-2z}$ as $z\to-\infty$. Throughout this layer, $\psi = -\frac{1}{3}+\frac{1}{3}(p'+1)(y+1)$ and so the equation satisfied by θ is

$$\frac{\partial^2\theta}{\partial z^2}+2\frac{\partial\theta}{\partial z} = 0.$$

Thus a balance between conduction and convection at constant velocity is achieved in this layer. The solution for θ is

$$\theta = \pi^{-\frac{1}{2}}e^{-2z}+B(\tilde{x}).$$

Matching with (4.4) shows that $\delta^{\frac{1}{2}}e^{-\delta} = (\log\beta)^{-\frac{3}{2}}$ and $B(\tilde{x}) = (-p')^{\frac{1}{2}}\pi^{\frac{1}{2}}k/2\sqrt{2}$.

The structure of the boundary layers when $\tilde{x} = O(1)$ is illustrated in figure 3.

The similarity variable η is given by

$$y = -1+\frac{\beta^{-1}\log\beta}{(-p')(p'+1)}(\eta+6)$$

so the solution breaks down when

$$-p'(p'+1) = O(\beta^{-1}\log\beta).$$

As $\tilde{x}\to-\infty, p'+1 = O(\exp[+\frac{1}{24}\pi\tilde{x}])$ from (4.15) and so we see that the solution will break down as $\tilde{x}\to(-24/\pi)\log\beta$. To match properly with the earlier flow we would therefore need another internal scaling for $\tilde{x}$. However, since we have been unable to determine the earlier flow analytically there is nothing to be gained by such detailed matching procedures.

As $\tilde{x}\to+\infty$, we see from (4.15) that $p' \sim -(\frac{1}{24}\pi\tilde{x})^{-\frac{1}{2}}$ and the solution will therefore only break down in this direction when $\tilde{x} = O(\beta^2(\log\beta)^{-2})$, which is when $x = O(1)$.

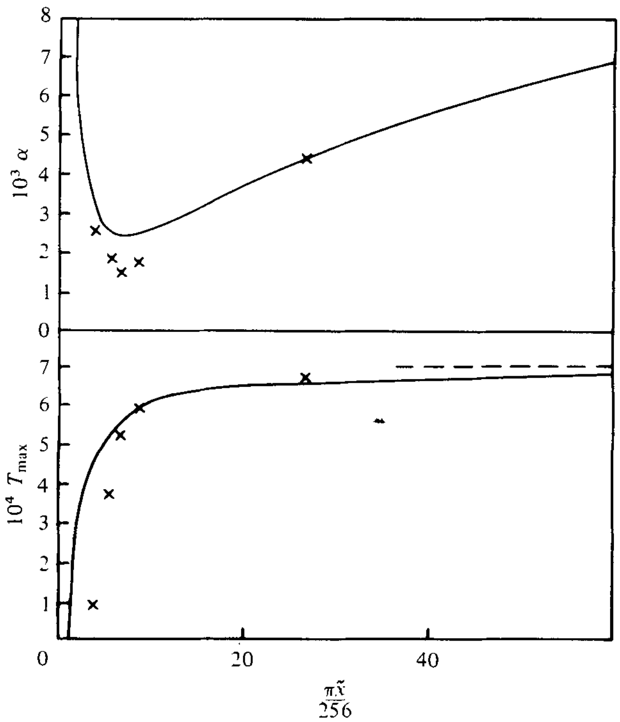

FIGURE 4. Comparison of numerical and analytic results: ——, asymptotic solution; ×, numerical results.

The solution as $\tilde{x} \to +\infty$ contains the similarity solution obtained in §4.1 of the paper by Pearson (1977).

5. Final remarks

The matching arguments used in this solution are self-consistent as far as they go. There is a gap in the matching in the boundary layer when $\bar{x} = O(1)$ and the full solution of equations (3.5) is relevant. Our solutions are physically plausible and since they describe a dramatic stage in the flow development, where the pressure gradient changes rapidly, may be of some interest.

The analysis predicts a pressure gradient with a point of inflexion and as a result the thermal boundary layer width first decreases and then increases as x increases. From the expression given in (4.10), the maximum value for T during the transition phase is

$$T_{\max} = \beta^{-1}\log\beta(1+2C/\log\beta) = \beta^{-1}\log\left(\tfrac{1}{72}\beta(p'+1)^2\right)$$

on substituting for C. Thus as $p' \to 0$, the maximum value of T approaches the fixed value $\beta^{-1}\log\left(\frac{1}{72}\beta\right)$.

An exactly analogous solution can be obtained for flow along an asixymmetric pipe. The pressure gradient (again normalized with respect to its value at the inlet) satisfies equation (4.15) with $\frac{1}{24}\tilde{x}$ replaced by $\frac{1}{256}\tilde{x}$.

A numerical solution of the full problem for axisymmetric flow with $\beta = 64 \times 10^4$

has been obtained by C. A. Heiber of Cornell University. This solution exhibits the qualitative features predicted by the asymptotic analysis. A comparison of the values of $T_{\max}$ and D, the non-dimensional distance of this maximum from the wall, found from the numerical solution of the full equations and from the asymptotic solution is shown in figure 4. The discrepancies can to some extent be accounted for by the fact that $\log \beta = 13{\cdot}37$ is not really very large in this case.

Appendix

The relationships between the variables used here and those used by Pearson (1977) are listed below.

	This paper	Pearson's paper
Channel width	$2d$	h
Non-dimensional transverse co-ordinate	y	2η
Non-dimensional downstream co-ordinate	x	$\dfrac{4}{3P_e}\xi$
Non-dimensional stream function	ψ	$\frac{2}{3}\phi$
Reference velocity	U	$3V$
Non-dimensional pressure gradient	p'	$\frac{1}{12}\pi_\xi$
Non-dimensional temperature	T	$\frac{1}{9}\theta$
Nahme–Griffith number	β	$9G$
Flow rate Q	$\frac{2}{3}Ud$	Vh

REFERENCES

FUJITA, H. 1969 *Bull. Am. Math. Soc.* **75**, 132.

OCKENDON, H. & OCKENDON, J. R. 1977 *J. Fluid Mech.* **83**, 177.

PEARSON, J. R. A. 1977 *J. Fluid Mech.* **83**, 191.